

# Stable Black Strings in Anti-de Sitter Space

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## Abstract

In the five-dimensional Einstein gravity with negative cosmological constant in the presence/absence of a *non-fine-tuned* 3-brane, we have investigated the classical stability of black string solutions which are foliations of four-dimensional  $AdS/dS$ -Schwarzschild black holes. Such black strings are generically unstable as in the well-known Gregory-Laflamme instability. For  $AdS$  black strings, however, it turns out that they become stable if the longitudinal size of horizon is larger than the order of the  $AdS_4$  radius. Even in the case of unstable black strings, the  $AdS$  black strings have a very different feature of string fragmentations from that in the flat brane world. Some implications of our results on the Gubser-Mitra conjecture are also discussed.

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## I. INTRODUCTION

The Schwarzschild black hole which has a compact black hole horizon is known to be stable in general relativity [1]. However, it has been known that black string/brane solutions which are a foliation of lower dimensional Schwarzschild black holes are unstable, the so-called Gregory-Laflamme instability [2]. This instability extends to a much broader range of charged black branes in string theory with the exception for extremal or near extremal cases [3,4].

In the presence of a cosmological constant, Gregory [5] has for the first time shown that this black string instability persists to hold. Recently, Randall and Sundrum [6] have proposed an interesting model where the gravity in higher dimensions with negative cosmological constant is localised on a lower dimensional domain wall. In the flat brane world model where the tension of a 3-brane is fine-tuned with the five-dimensional cosmological constant, any Ricci-flat four-dimensional metric can be embedded. For instance, a black string solution, which is simply a foliation of four-dimensional Schwarzschild black holes perpendicular to the 3-brane, can be easily constructed. The properties of such black string were investigated in Refs. [7,8]. In particular, they argued that the black string is unstable near the  $AdS_5$  horizon, but becomes stable in the vicinity of the 3-brane, indicating the fragmentation into a cigar type (or a pancake-like more accurately [8]) black hole across the brane as a final state. Gregory [5] has confirmed this conjecture by explicitly performing the linearized perturbation analysis of the black string background in this flat brane world scenario. The author also showed the instability of a black string embedded in the  $AdS_5$  spacetime without the 3-brane.

As long as we know, however, all black string/brane instabilities mentioned above were shown within the context of asymptotically locally flat spacetimes in the sense that all slices orthogonal to the string or brane are asymptotically flat.<sup>1</sup> Actually, in the presence of a negative cosmological constant, it is also possible to construct black string solutions which are a foliation of black holes that are asymptotically *non-flat*. For example, the pure  $AdS_5$  spacetime can be sliced into pure four-dimensional anti-de Sitter ( $AdS_4$ ) or de Sitter ( $dS_4$ ) spacetimes. Then, it is straightforward to show that any four-dimensional metric satisfying four-dimensional Einstein equations with the same cosmological constant can be embedded into  $AdS_5$  space. In particular, the  $AdS_4$  ( $dS_4$ )-Schwarzschild black hole can be embedded, resulting in a five-dimensional hypercylindrical  $AdS$  ( $dS$ ) black string solution. Similarly, even in the curved brane world models where a 3-brane with non-fine-tuned tension is introduced in  $AdS_5$  backgrounds [10,11], this generalisation holds (see Ref. [12] for the  $dS$  embedding.).

Now it will be of interest to see whether the Gregory-Laflamme instability still persists to hold for these black string solutions in  $AdS$  backgrounds. It is because some naive arguments

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<sup>1</sup>Actually Emparan, Horowitz, and Myers have argued, based on entropy comparison, the stability of ‘BTZ’ black string solutions in *four-dimensional*  $AdS$  space [9]. However, they put two 2-branes with a suitable identification of spacetimes. It seems that, even if the separation of two 2-branes is taken to be infinitely large, this black string does not correspond to a one dimension less configuration of our  $AdS$  case with a single 3-brane due to the specific embedding of the 2-branes.

given below seem to indicate that the stability behavior for such black strings could be very different from the known Gregory-Laflamme instability. In addition, recently Gubser and Mitra [13] have proposed an interesting conjecture about the relationship between the classical black string/brane instability and the local thermodynamic stability [4]. It states that a black string/brane with a non-compact translational symmetry is classically stable if, and only if, it is locally thermodynamically stable. Since black string solutions in  $AdS$  space have warped geometries along the extra dimension, our study will show what will happen when the assumption of translational symmetry is discarded in the Gubser-Mitra (GM) conjecture.

In this paper, we investigate the stability of black string solutions which are asymptotically locally  $AdS_4/dS_4$  mentioned above. It turns out that such black strings are generically unstable as usual. Interestingly  $AdS$  black strings, however, become stable if the four-dimensional horizon radius is larger than the order of  $AdS_4$  radius. This stability can be understood naively since the geometry along the string produces a sort of an effective compactification whose scale is determined by the  $AdS_4$  radius. Even in the case of unstable  $AdS$  black strings, the feature of instability is very different near the conformal infinity. In fact, they become stable near the boundary of  $AdS_5$ . We will first present some naive arguments indicating this behavior by using entropy comparisons and the Gubser-Mitra conjecture. Then, linearized metric perturbation analysis and numerical results are shown explicitly. Finally, some discussions and physical implications of our results are followed.

## II. BLACK STRING SOLUTIONS AND NAIVE STABILITY ARGUMENTS

Let us consider the five-dimensional pure anti-de Sitter spacetime whose metric is given in the following form [10,11]

$$ds^2 = H^{-2}(z)(\gamma_{\mu\nu}dx^\mu dx^\nu + dz^2). \quad (1)$$

Here the warping factors are

$$dS_4(\Lambda_4 > 0) : \quad H(z) = l_4/l_5 \sinh z/l_4 \quad (2)$$

$$M_4(\Lambda_4 = 0) : \quad H(z) = z/l_5 \quad (3)$$

$$AdS_4(\Lambda_4 < 0) : \quad H(z) = l_4/l_5 \sin z/l_4, \quad (4)$$

where  $\Lambda_5 = -6/l_5^2$  and the four-dimensional cosmological constant  $\Lambda_4 = \pm 3/l_4^2$  is arbitrary. The metric  $\gamma_{\mu\nu}$  describes four-dimensional de Sitter, flat Minkowski, and anti-de Sitter spacetimes, respectively, depending on the warping factors. These metrics actually describe the same five-dimensional anti-de Sitter spacetime with radius  $l_5$ , and simply correspond to different ways of slicing it. If we introduce a 3-brane with uniform tension  $\sigma$  at  $z = 0$  perpendicular to the fifth direction, one needs cutting and gluing of parts of the  $AdS_5$  spacetime in order to make the geometry smooth around the 3-brane. The resulting geometries are still described by the metrics above with replacing  $z \rightarrow |z| + c$ . Here  $c$  is an arbitrary integral constant which is related to the location of a 3-brane. This  $c$  and the tension of a 3-brane determine the cosmological constant  $\Lambda_4$ ;  $|\sigma| = \sigma_0 \cosh c/l_4$  for  $dS$  brane,  $|\sigma| = \sigma_0 = 3\sqrt{-\Lambda_5/6}/8\pi G_5$  for flat brane, and  $|\sigma| = \sigma_0 \cos c/l_4$  for  $AdS$  brane [10,11].

Now the Ricci-flat metric embedding in the flat brane world can be generalized as follows: The metric given in Eq. (1) satisfies the five-dimensional Einstein equation with a negative cosmological constant  $\Lambda_5$  if the metric  $\gamma_{\mu\nu}$  is any solution of the four-dimensional Einstein equation with the cosmological constant  $\Lambda_4$ . In particular, one can easily construct  $AdS/dS$  black string solutions by taking the  $AdS_4/dS_4$ -Schwarzschild black holes for  $\gamma_{\mu\nu}$  such as

$$ds^2 = H^{-2}(z) \left[ -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_2^2 + dz^2 \right], \quad (5)$$

where

$$f(r) = 1 - \frac{r_0}{r} - \frac{\Lambda_4}{3}r^2. \quad (6)$$

We wish to study the classical stability of black string solutions constructed in Eq. (5) with the warping factors  $H(z)$  in Eqs. (2-4) under linearized metric perturbations. Especially, we will focus on the case of  $AdS$  black strings because their nature of stability is quite different from those of other two cases and usual black strings studied in Refs. [2,3,5,4]. Before going into the linearized analysis in detail, let us give several naive arguments revealing the basic nature of stabilities in considerations.

The five-dimensional Riemann tensor squared of black strings in Eq. (5) is given by

$$R_{MNPQ}(g)R^{MNPQ}(g) \simeq \frac{10}{9}\Lambda_5^2 + H^4(z) \left[ R_{\mu\nu\alpha\beta}(\gamma)R^{\mu\nu\alpha\beta}(\gamma) - \frac{8}{3}\Lambda_4^2 \right], \quad (7)$$

where  $R_{\mu\nu\alpha\beta}(\gamma)$  is the four-dimensional Riemann tensor constructed from the metric  $\gamma_{\mu\nu}$ . Thus, for the cases of  $dS$ -Schwarzschild black strings (2) and Schwarzschild black strings (3), there generically exist curvature singularities at  $z = \infty$  in addition to the usual singularity at the center of black strings (*e.g.*,  $r = 0$ ). In fact, these are naked singularities, not surrounded by some event horizon [7]. For Schwarzschild black strings in particular, Gregory [5] has shown that they are unstable, presumably indicating fragmentation into an array of black holes. If these black strings were stable and so no tendency to fragmentation, the full 5D spacetimes would be pathological due to such naked singularities. For  $AdS$  black strings (4), however, the function  $H(z)$  is finite everywhere and so there is no naked or curvature singularity other than the usual ones at  $r = 0$ . Therefore, we expect that  $dS$  black strings would be unstable at least near the ‘‘Rindler horizon’’  $z = \infty$ , but  $AdS$  black strings would not necessarily have to be so.

A common and more convincing argument is based on entropy comparison between different black hole configurations. The existence of black string instability is often explained by arguing that there exists a length for a segment of black string above which a compact black hole with the same mass becomes entropically favourable [2,3]. It possibly indicates that a black string decays into an array of black holes. We will compare the entropy contained in a segment of the black string with that contained in a five-dimensional compact black hole of the same mass. Since there is no known exact five-dimensional black hole solution in the presence of a 3-brane, we consider only the case with no 3-brane. However, we believe the result obtained equally applies to the case with a 3-brane because the presence of a 3-brane does not change the important nature of the Kaluza-Klein mass spectrum relevant in the linearized perturbation analysis as shall be shown below.

The entropy contained within a segment of  $AdS/dS$  black string can be obtained by integrating the area of the horizon as follows,

$$S_{\text{b.s.}} = \frac{A}{4} = \frac{1}{4} \int_a^b \mathcal{A}(z) \frac{dz}{H(z)} = \pi r_+^2 L, \quad L = \int_a^b \frac{dz}{H^3(z)}. \quad (8)$$

Here  $\mathcal{A}(z) = 4\pi r_+^2(z) = 4\pi r_+^2/H^2(z)$  with  $f(r_+) = 0$  is the area of the  $AdS_4/dS_4$ -Schwarzschild black hole measured by an observer at  $z = \text{const.}$ . Notice that we can set  $l_4 = l_5 = l$  by using diffeomorphism. The mass contained in the segment can also be obtained by integrating the first law of black hole thermodynamics  $\delta M = T\delta S$  as in Ref. [8]

$$M = \int_0^{r_+} T \frac{\partial S}{\partial r_+} dr_+ = \frac{r_0 L}{2}. \quad (9)$$

Now the *five*-dimensional  $AdS$ -Schwarzschild black hole is described by

$$ds^2 = -f(R)dt^2 + \frac{1}{f(R)}dR^2 + R^2 d\Omega_3^2, \quad f(R) = 1 + \frac{R^2}{l^2} - \frac{R_0^2}{R^2}. \quad (10)$$

Thus, black hole entropy and mass become

$$S_{\text{b.h.}} = \frac{\pi}{3} R_+^3, \quad M = \frac{R_0^2}{4}, \quad (11)$$

respectively. By identifying the mass with that of the string segment, one can express  $\Delta S = S_{\text{b.s.}} - S_{\text{b.h.}}$  as a function of  $r_+$ ,  $l$ , and  $L$ . Surprisingly, this difference can be positive, independent of the “length” of black string segment  $L$  provided that

$$dS(\Lambda_4 > 0) : \quad \frac{\sqrt{13}-3}{2}l \simeq 0.30l < r_+, \quad (12)$$

$$AdS(\Lambda_4 < 0) : \quad \frac{3-\sqrt{5}}{2}l \simeq 0.38l < r_+ < \frac{3+\sqrt{5}}{2}l \simeq 2.62l. \quad (13)$$

Note that, for the case of  $dS$  black string, the event horizon should be inside of its cosmological horizon,  $r_+ \leq l/\sqrt{3} \simeq 0.58l$  (*i.e.*,  $r_0 \leq 2l/3\sqrt{3} \simeq 0.38l$ ).

Thus, it appears that black strings are entropically more favourable than the five-dimensional  $AdS$ -Schwarzschild black hole with the same mass no matter how long the hypercylindrical horizon of black string is extended, possibly indicating stability, if the size of four-dimensional horizon  $r_+$  lies on the range shown above. For black strings with the horizon radius  $r_+$  not belonging to this range, the black hole becomes entropically favourable as the “length” of string segment  $L$  increases as usual. Interestingly, however, the black string segment again becomes entropically favourable if its “length” increases further. Here, however, we would like to point out that this sort of “global” thermodynamic stability argument should not be taken seriously since this argument in the viewpoint of the classical black hole area theorem shows only some plausibility for the classical decay of black strings.

The Gubser-Mitra [13] conjecture can be regarded as a refinement of the “global” entropy argument given above, and is proved in Ref. [4]. Although black string solutions we consider do not have a translational symmetry due to the warping factors in Eqs. (2-4), it will be interesting to apply this conjecture to our case. The local thermodynamic stability of a

segment of  $AdS/dS$  black string will be determined by the sign of the heat capacity given by

$$\frac{dM}{dT} = -2\pi \frac{1 - \Lambda_4 r_+^2}{1 + \Lambda_4 r_+^2} r_+^2 L. \quad (14)$$

For the  $AdS$  case, one can easily see that the heat capacity is negative for  $r_+ < 1/\sqrt{-\Lambda_4} = l_4/\sqrt{3}$ , but becomes positive for  $r_+ > l_4/\sqrt{3}$  [14]. Thus, we expect  $AdS$  black strings become stable classically when  $r_+ > l_4/\sqrt{3}$  according to the GM conjecture. On the other hand,  $dS$  black strings are expected to be unstable classically since they are locally thermodynamically unstable for  $r_+ < l_4/\sqrt{3}$  and the cosmological horizon locates at  $r_+ = l_4/\sqrt{3}$ .

### III. LINEARIZED PERTURBATION ANALYSIS

So far, we have given three naive arguments which possibly indicate that  $AdS$  black strings are stable when the four-dimensional horizon radius becomes large. Now let us perform the classical stability analysis explicitly. We consider small metric perturbations about  $AdS/dS$  black string background spacetimes and see whether or not there exists any mode which is regular spatially, but grows exponentially in time. By choosing the Randall-Sundrum gauge [5,6,15], *vacuum* metric perturbations can be written as follows

$$ds^2 = H^{-2}(z) \left[ (\gamma_{\mu\nu} + h_{\mu\nu}) dx^\mu dx^\nu + dz^2 \right], \quad \nabla^\mu h_{\mu\nu} = 0, \quad h = \gamma^{\mu\nu} h_{\mu\nu} = 0, \quad (15)$$

where  $\nabla$  is the covariant derivative operator compatible with the four-dimensional  $AdS/dS$ -Schwarzschild black hole metric  $\gamma_{\mu\nu}(x)$  in Eq. (5). Then the linearised Einstein equations for vacuum metric fluctuations become simply

$$\square h_{\mu\nu}(x, z) + 2R_{\mu\rho\nu\tau} h^{\rho\tau}(x, z) - \left( -\partial_z^2 + \frac{3\partial_z H}{H} \partial_z \right) h_{\mu\nu}(x, z) = 0, \quad (16)$$

where  $\square \equiv \gamma^{\rho\tau} \nabla_\rho \nabla_\tau$  and  $h^{\mu\nu} \equiv \gamma^{\mu\rho} \gamma^{\nu\tau} h_{\rho\tau}$ . Putting  $h_{\mu\nu}(x, z) = H^{3/2}(z) \xi(z) h_{\mu\nu}(x)$ , this equation can be decomposed into the four dimensional part (*e.g.*, massive Lichnerowicz equation) and the fifth part as usual

$$\Delta_L h_{\mu\nu}(x) \equiv \square h_{\mu\nu}(x) + 2R_{\mu\rho\nu\tau} h^{\rho\tau}(x) = m^2 h_{\mu\nu}(x), \quad (17)$$

$$[-\partial_z^2 + V(z)] \xi(z) = m^2 \xi(z), \quad V(z) = -\frac{3}{2} \frac{H''}{H} + \frac{15}{4} \left( \frac{H'}{H} \right)^2. \quad (18)$$

Thus, one sees that essentially the fifth dimension gives massive gravitons as usual in the Kaluza-Klein (KK) point of view and their mass spectrum can be read off from the form of effective potential  $V(z)$ . For the flat case in Eq. (3),  $V(z)$  vanishes as  $z \rightarrow \infty$  and so the KK mass spectrum is continuous starting at  $m = 0$ . For the  $dS$  case in Eq. (2),  $V(z)$  goes to a non-zero constant,  $9/4l_4^2$ , and the KK mass spectrum is again continuous, but has a non-zero minimum mass,  $m_{\min.} = 3/2l_4$ . For the  $AdS$  case, however,  $V(z)$  grows infinitely, making effectively a confining box due to  $AdS$  nature of the spacetime. Thus, the  $z$  direction is effectively compactified even if it is still infinite in proper length. Consequently, the KK mass spectrum becomes discrete now and its lowest mass is  $m_{\min.} = 4/l_4$ . Here one can

observe that the scale of effective compactification is  $l_4$  instead of  $l_5$ . Note that this effective compactification does not happen when the  $AdS_5$  spacetime is sliced into four-dimensional Minkowski or  $dS_4$  submanifolds. When a 3-brane is introduced, it gives a delta-function-like potential well at the position of the 3-brane, producing an attractive force. Hence the KK mass spectrum for black string in brane world scenarios has basically the same feature as that in the case of no 3-brane, but its magnitude is somewhat reduced [10,16]. Here we should point out that the non-zero finiteness of the lowest KK mass for  $AdS/dS$  cases plays an important role for the stability of black strings as shall be shown below explicitly.<sup>2</sup>

As explained above, we wish to find any instability mode which is a solution of the massive Lichnerowicz equation in Eq. (17) with suitable reference to gauge and boundary conditions at the future event horizon and the spatial infinity. Since higher angular momentum fluctuation modes are more stable in general, we will consider a zero angular momentum mode only, an  $s$ -wave mode [2,3]. General spherically symmetric perturbations which give instability can be written in canonical form as [1,2]

$$h_{\mu\nu}(x) = e^{\Omega t} \begin{pmatrix} H_{tt}(r) & H_{tr}(r) & 0 & 0 \\ H_{tr}(r) & H_{rr}(r) & 0 & 0 \\ 0 & 0 & K(r) & 0 \\ 0 & 0 & 0 & K(r) \sin^2 \theta \end{pmatrix}, \quad (19)$$

with  $\Omega > 0$ . Using transverse traceless (TTF) gauge conditions in Eq. (15), we can eliminate all but one variable, say  $H_{tr}$ , from the Lichnerowicz equation, obtaining a second order ordinary differential equation as follows [3,17]:

$$\begin{aligned} & \left[ \Omega^2 + m^2 f + \frac{f f''}{2} - \frac{f'^2}{4} + \frac{f f'}{r} \right] H_{tr}'' \\ & + \left[ \frac{\Omega^2}{f} \left( 3f' + \frac{2f}{r} \right) + m^2 \left( 2f' + \frac{2f}{r} \right) + \left( \frac{3f' f''}{2} + \frac{f f''}{r} + \frac{3f'^2}{2r} - \frac{3f'^3}{4f} + \frac{2f f'}{r^2} \right) \right] H_{tr}' \\ & + \left[ - \left( \frac{\Omega^2}{f} + m^2 \right)^2 + \frac{\Omega^2}{f} \left( -\frac{2f}{r^2} + \frac{f'}{r} + \frac{f''}{2} + \frac{5f'^2}{4f} \right) + m^2 \left( -\frac{2f}{r^2} + \frac{2f'}{r} + \frac{f''}{2} - \frac{f'^2}{4f} \right) \right. \\ & \quad \left. + \left( \frac{5f'^2}{2r^2} - \frac{f'^3}{r f} + \frac{f'^2 f''}{4f} - \frac{f f''}{r^2} + \frac{3f' f''}{r} + \frac{f'^2}{2} - \frac{2f f'}{r^3} - \frac{f'^4}{4f^2} \right) \right] H_{tr} = 0. \end{aligned} \quad (20)$$

Asymptotically  $H_{tr}$  have the following solutions;

$$H_{tr} \sim \begin{cases} r^{-5/2 \pm \sqrt{9/4 + (m l_4)^2}} & \text{for } r \rightarrow \infty, \\ (r - r_+)^{-1 \pm \Omega/2\kappa} & \text{for } r \rightarrow r_+, \end{cases} \quad (21)$$

for the  $AdS$  case, and

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<sup>2</sup>Actually, there exist massless KK modes as well for both  $AdS$  and  $dS$  cases. They are not normalizable except for the  $dS$  case with a 3-brane. However, these massless modes are irrelevant for searching instability modes as will be explained in more detail below.

$$H_{tr} \sim \begin{cases} (r_{++} - r)^{-1 \pm \Omega/2\kappa_{++}} & \text{for } r \rightarrow r_{++}, \\ (r - r_+)^{-1 \pm \Omega/2\kappa} & \text{for } r \rightarrow r_+, \end{cases} \quad (22)$$

for the  $dS$  case. Here  $\kappa = f'(r_+)/2$  and  $\kappa_{++} = -f'(r_{++})/2$  are surface gravities of the event and cosmological horizons, respectively, and  $r_{++}$  in the  $dS$  case denotes the cosmological horizon. As emphasized in Refs. [1–3], it is very important to impose right boundary conditions on the perturbations. Since our analysis is based on linearized equation, any fluctuations should remain “small.” This seemingly excludes asymptotic solutions with negative roots near the horizon. However, one can see that even asymptotic solutions with positive root diverge near the horizon when  $\Omega/2\kappa < 1$ . It is probably due to that the Schwarzschild coordinates are not good near the horizon. In fact, it turns out that, if we use some regular coordinate system such as Kruskal coordinates, the only asymptotic solution suitable for our linearized analysis is the one with positive root with any  $\Omega$  ( $> 0$ ) as pointed out in Refs. [2,3]. At the spatial infinity for the  $AdS$  case, if we require vanishing boundary condition as usual [18], even the positive root satisfies this condition provided  $ml_4 < 2$ . Actually, asymptotic  $AdS$  spacetimes are not globally hyperbolic. Thus, one needs to impose some extra condition by hand in order to make the dynamics of metric fluctuations well posed [19]. As imposed usually for matter fields in the pure  $AdS$  background spacetime [20], we require the total energy of gravitational fluctuations on this  $AdS$  background should be conserved. This requirement is satisfied only if  $\lambda < -5/2$  when  $H_{tr} \sim r^\lambda$  at  $r \sim \infty$ . Hence only the negative root is satisfactory in Eq. (21). We will give the details in [17].

With these boundary conditions described above, we now search instability modes characterized by  $(\Omega, m)$  which are solutions of Eq. (20) for given  $r_0$  and  $l_4$ . In other words, for  $AdS$  case we start from a solution with  $H_{tr} \sim r^{-5/2 - \sqrt{9/4 + (ml_4)^2}}$  at  $r \sim \infty$ , and find  $\Omega$  which makes  $H_{tr}$  extrapolate to  $H_{tr} \sim (r - r_+)^{-1 + \Omega/2\kappa}$  near the horizon through Eq. (20). Similarly, we start from  $H_{tr} \sim (r_{++} - r)^{-1 + \Omega/2\kappa_{++}}$  near the cosmological horizon for the  $dS$  case. Since the equation Eq. (20) is quite complicate to handle analytically, we solve it numerically. It, however, is worth to observe some scaling symmetries in Eq. (20) as follows:

$$r \rightarrow \alpha r, \quad r_0 \rightarrow \alpha r_0, \quad l_4 \rightarrow \alpha l_4 \ (\Lambda_4 \rightarrow \alpha^{-2} \Lambda_4), \quad \Omega \rightarrow \alpha^{-1} \Omega, \quad m \rightarrow \alpha^{-1} m. \quad (23)$$

So we can fix one of these parameters, say  $l_4 = 1$ . Moreover when  $r_0 \gg l_4$ ,  $f \sim -r_0/r \pm r^2/l_4^2$ , and so there exists another approximate scaling symmetry for large black holes given by [18]

$$r \rightarrow \alpha r, \quad r_0 \rightarrow \alpha^3 r_0, \quad l_4 \rightarrow l_4, \quad \Omega \rightarrow \alpha \Omega, \quad m \rightarrow m. \quad (24)$$

Then we find  $\Omega \sim r_0^{1/3}$  for the case of large black holes.

Using Mathematica and Gear method for solving differential equations, we obtained the results in Fig. 1. For both  $AdS/dS$  cases, one can see that the instability shrinks in parameter space as the mass parameter  $r_0$  increases with fixed  $AdS/dS$  radius. More precisely, the mass of the so-called threshold unstable mode [4],  $(\Omega, m) = (0, m_*)$ , decreases fast down to zero as the horizon radius  $r_+$  increases for the  $AdS$  case whereas it approaches to some non-zero finite value as  $r_+$  increases towards the cosmological horizon for the  $dS$  case. Thus, it appears that there always exist instability modes. However, the KK mass  $m$  cannot be arbitrary, but is determined by the geometry in the fifth direction through Eq. (18) as explained before. The lowest KK masses are  $4/l_4$  and  $3/2l_4$  for the  $AdS$  and



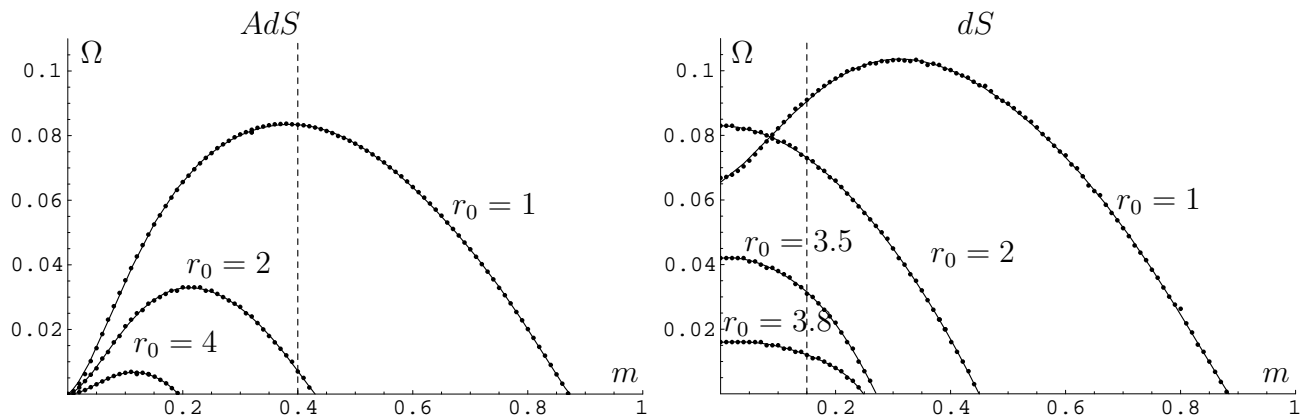


FIG. 1. The left figure is for the  $AdS$  case with  $r_0 = 1, 2$  and  $4$ . The right figure is for the  $dS$  case with  $r_0 = 1, 2, 3.5$ , and  $3.8$ . The Nariai solution corresponds to  $r_0 \simeq 3.85$ . The fixed  $AdS$  and  $dS$  radius is  $l_4 = 10$ . The straight vertical lines denote the lowest KK masses,  $0.4$  for  $AdS$  and  $0.15$  for  $dS$ .

$dS$  cases without 3-brane, respectively, as denoted in Fig. 1 by straight vertical lines. So, if the threshold mass becomes smaller than this lowest KK mass for a certain  $r_0$ , there exists indeed no unstable mode. As can be seen in Fig. 1, this happens actually when  $0.21l_4 \lesssim r_0$  (*i.e.*,  $0.20l_4 \lesssim r_+$ ) for the  $AdS$  case. Therefore, we find that the  $AdS$  black string is unstable when its four-dimensional horizon size is small, but it becomes stable when the horizon size is larger than the order of the  $AdS_4$  radius (*i.e.*,  $r_+^{\text{cr}} \simeq 0.20l_4$ ). On the other hand, the presence of a 3-brane reduces the lowest KK mass. Consequently, it increases the value of critical horizon radius for stable black strings in  $AdS$  brane world model. In particular, in the vicinity of the flat brane world (*i.e.*,  $\Lambda_4 \sim 0$  or  $l_4 \sim \infty$ ),  $AdS$  black strings become almost always unstable since  $m_{\text{min.}} \simeq 0$ , which can be expected from the results in Ref. [10].

For the  $dS$  case, on the other hand, although the threshold mass decreases as the horizon radius increases up to the cosmological one, they all still seem to remain larger than the lowest KK mass (see also Fig. 3). Therefore,  $dS$  black strings seem to be always unstable. In particular, the instability seems to persists all the way down to the Nariai solution in which the event horizon coincides with the cosmological horizon. It, however, should be pointed out that the Nariai limit must be treated separately since boundary conditions become invalid and the numerical error in our analysis increases near such extremal case. As argued in Ref. [12], the stability behavior of this case might be very different from that of non-extremal cases. The presence of a 3-brane in  $dS$  black strings again makes the system more unstable since it reduces the lowest KK mass in the unit of  $l_4$ . For the flat case (*i.e.*,  $\Lambda_4 = 0$ ), we have confirmed the results obtained in Ref. [5]. That is, since the KK mass spectrum is continuous with zero lowest mass and the threshold mass asymptotes to zero as  $r_0 \rightarrow \infty$  as can be seen in Fig. 3, all black strings are unstable in this case.

Fig. 2 illustrates how the threshold mass changes for a given  $r_0$  as the cosmological constant  $\Lambda_4$  varies away from zero. It shows that the instability in parameter space shrinks as  $\Lambda_4$  becomes negative (*i.e.*,  $AdS$  case), but expands as  $\Lambda_4$  becomes positive (*i.e.*,  $dS$  case). In other words, basically adding negative cosmological constant has a stabilising effect as in the case of adding charge into black strings [3] whereas adding positive cosmological constant gives destabilising influence.

It will be interesting to see how well the results obtained by explicit perturbation analysis

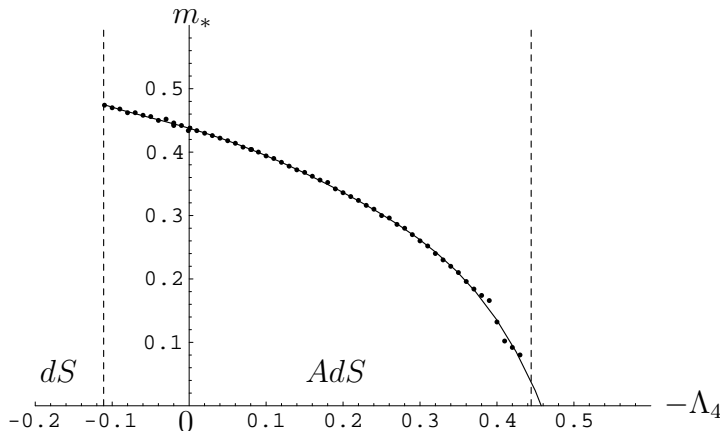


FIG. 2. The threshold masses  $m_*$  for varying  $\Lambda_4$  with given  $r_0 = 2$ . The left vertical dotted line denotes the Nariai limit and the right one the critical  $\Lambda_4$  predicted by the GM conjecture.

agree with those in naive arguments given before. For the  $AdS$  case, critical values for stable black strings were predicted as  $r_+ \simeq 0.38l_4, 0.58l_4$  in Eqs. (13) and (14) by the entropy comparison and by the GM conjecture, respectively. The numerical results predict  $r_+ \simeq 0.20l_4$  which agrees within the order of one. The entropy comparison, however, also predicts another critical horizon radius,  $r_+ \simeq 2.62l_4$ , across which black strings become unstable again. We have searched various parameters around this critical value, but could not find any unstable black string. Thus, our numerical results agree well, at least qualitatively, with the GM conjecture, but with the entropy argument only in part. For the  $dS$  case, on the other hand, only the prediction in GM conjecture agrees well with the numerical results.

Fig. 3 shows how the threshold mass for a given  $\Lambda_4$  decreases as the black hole becomes large. These numerical results agree qualitatively well with those in Refs. [21] obtained analytically with some approximation and different gauge choices in a different context. Both flat (*i.e.*,  $\Lambda_4 = 0$ ) and  $AdS$  (*i.e.*,  $\Lambda_4 < 0$ ) cases give almost same decreasing pattern for small  $r_0$ , but they start to deviate as the black hole becomes large, around  $r_0 \simeq 4$ . As  $r_0 \rightarrow \infty$  for the flat case,  $m_*$  denoted by the dotted curved line in Fig. 3 asymptotes to zero ( $\sim 1/r_0$ ), but never touches it. Consequently, since the continuum KK mass spectrum starts at  $m = 0$ , one can see again that all black strings are unstable no matter how large  $r_0$  is. If the black string is compactified, however, the continuum KK mass spectrum becomes discrete. The massless mode is not a real instability mode, but presumably a gauge artifact [3,2] since the Lichnerowicz equation with  $m = 0$  becomes that of pure four-dimensional black holes. So, the lowest instability mode will start at non-zero  $m$ . Then Fig. 3 shows that compactified black strings in the flat case will become stable if  $r_0$  is larger than some critical value determined by the compactification scale. The stability of  $AdS$  black strings can be understood similarly in this point of view. It is because the  $AdS_5$  nature of geometry in the fifth direction with  $AdS_4$  slicing gives an effective compactification whose scale is determined by  $l_4$  instead of  $l_5$  as explained above. However, we point out there is another interesting feature in this case. As can be seen in Fig. 3 for the  $AdS$  case, although numerical error increases as  $m_*$  becomes small, the curve for  $m_*$  seems to touch the horizontal axis if the data points are extrapolated further. Moreover, this terminating point seems to agree with the critical value  $r_0 \simeq 0.77l_4$  (*i.e.*,  $r_+ \simeq 0.58l_4$ ) obtained by the GM conjecture, the horizon radius across which the heat capacity changes its sign. Consequently, one can expect that

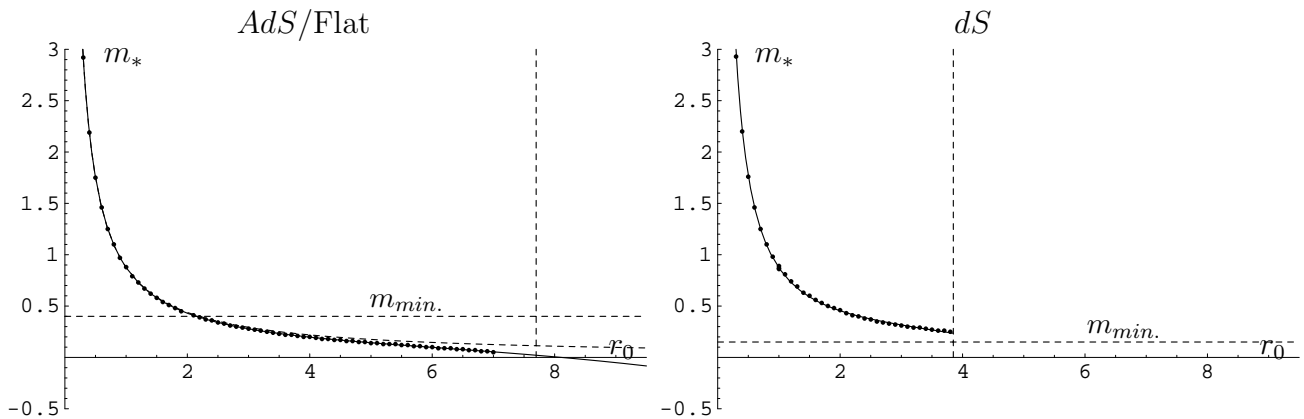


FIG. 3. The left figure: threshold masses for varying  $r_0$  with given  $l_4 = 10$  in the  $AdS$  case. The vertical dotted line denotes the critical  $r_0 \simeq 7.7$  predicted from the Gubser-Mitra conjecture. The numerical data stops at  $r_0 \simeq 7.0$ . The right figure: same diagram for the  $dS$  case. The vertical dotted line denotes the Nariai limit,  $r_0 \simeq 3.85$ . Note that  $m_*(r_0 \simeq 3.85) = 0.29 > m_{\min.} \simeq 0.15$ .

the black string will be stable at least if  $r_0$  is larger than this terminating value, no matter what the KK mass spectrum is. Therefore, in addition to the stabilization due to an effective compactification,  $AdS$  black string solutions seem to have a sort of intrinsic tendency for stabilization probably due to the  $AdS_4$  nature of longitudinal four-dimensional geometries.

It can be noticed that the critical value,  $r_+ \simeq 0.20l_4$ , obtained for  $AdS$  black strings does not exactly agree with, but occurs a bit “earlier” than that of the GM conjecture,  $r_+ \simeq 0.58l_4$ . This discrepancy, however, is expected because the GM conjecture assumes non-compact translational symmetry. Actually, this condition can be easily replaced in the proof of the GM conjecture [4] as follows: a black string/brane, as long as its KK mass spectrum is continuous starting at zero mass, is classically stable if, and only if, it is locally thermodynamically stable. One can see that the flat case satisfies this modified GM conjecture. The GM conjecture then predicts that a terminating point must exist if the system is locally thermodynamically stable. In fact, our numerical results show not only that the terminating point exists, but also that it agrees with the critical value of local thermodynamic stability in the GM conjecture. Thus, it can be expected that a black string having discrete KK mass spectrum becomes stable, if it happens, before the terminating point as in the  $AdS$  case.

Finally, we have so far concentrated on the feature of stability with special emphasis on stable black string configurations in  $AdS_5$  spacetimes. Now it will be also interesting to see what the final states would be for unstable black strings. In order to answer this question, we just need to know how the eigenfunction  $\xi(z)$  in Eq. (18) with given  $m$  behaves along the fifth coordinate  $z$ . For black strings in the flat case, it has been argued in Ref. [5] that the interval of successive wiggles in proper length becomes exponentially tiny towards the  $AdS_5$  horizon, and so the string is somewhat stable near the 3-brane but quickly becomes unstable away from it, generating an accumulation of “mini” black holes towards the  $AdS_5$  horizon. For  $dS$  case with 3-brane, the shape of the potential  $V(z)$  in Eq. (18) is a volcano type and similar to that of the flat case. The only difference is that  $V(z)$  approaches to a non-zero constant as  $z \rightarrow \infty$  (e.g., the “Rindler” horizon) instead of vanishing. Then  $\xi(z)$  will be similar to that of the flat case which is Bessel function, but goes to zero more quickly.

Accordingly, the feature of fragmentations will be almost same as that in flat case, with a bit stronger instability. For the  $AdS$  case, however, it turns out to be very different. The potential  $V(z)$  is again a volcano type around the 3-brane, but diverges at the boundary of  $AdS_5$  (*e.g.*, the conformal infinity), effectively creating a box. Thus,  $\xi(z)$  will behave like the Hermite function which is an eigenfunction of a harmonic oscillator with slight modifications in the vicinity of the 3-brane. Consequently, the black string becomes again stable near the boundary of  $AdS_5$  as well as in the vicinity of the 3-brane, generating multi black holes in between. This is why a segment of  $AdS$  black string becomes entropically favourable again when its length  $L$  in Eq. (8) becomes large enough.

#### IV. CONCLUSION

To conclude, we have shown that, although black strings in  $AdS$  spacetimes which are not locally asymptotically flat are generically unstable classically under linearized metric fluctuations, the  $AdS$  black string solutions are stable when the longitudinal size of the horizon is larger than the order of  $AdS_4$  radius. Generically, adding negative cosmological constant has a stabilization effect whereas adding positive cosmological constant has a destabilization influence. It will be straightforward to extend our study to higher dimensional cases. We believe the essential feature of stability for  $AdS$  black string/brane solutions in higher dimensions will be same.

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